5 - Existence of separatrices

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Results on separatrices

Our goal is to prove the following result

Thm (CAMACHO-SAD, 1982). Let $\mathcal{F}$ be a holomorphic foliation on $(\mathbb{C}^2, 0)$. Then $\mathcal{F}$ admits a (convergent) separatrix.

The strategy goes as follows.

1) Apply Seidenberg's theorem: $\exists\, \pi : X_\pi \to (\mathbb{C}^2, 0)$ composition of blow-ups s.t. the saturated lift $\hat{\mathcal{F}}_\pi$ of $\mathcal{F}$ at X_π has only reduced singularities.

2) Assume $\mathcal{F}$ is not dicritical

$\iff$ we don't blow-up any dicritical point in the reduction process

$\iff$ $\hat{\mathcal{F}}_\pi$ is tangent to $\pi^{-1}(0)$ the exceptional divisor

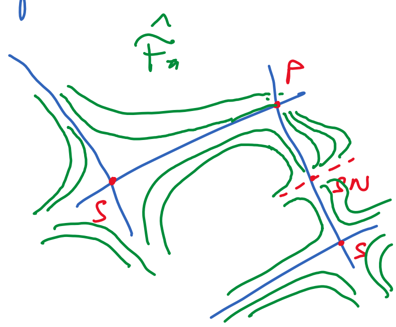
Rem: dicritical $\iff$ $\exists\ \infty$-many separatrices.

reduced singularities have either:

- 2 separatrices ($xy = 0$) if the index α is $\neq 0$ (using $\alpha \notin \mathbb{N}^* \cup \frac{1}{\mathbb{N}^*}$)
- 1 separatrix ($y = 0$) if $\alpha = 0$ (saddle-node).

A priori, these separatrices could coincide with the exceptional divisor.

We want to find a singularity $p \in \pi^{-1}(0)$ of $\hat{\mathcal{F}}_\pi$ that is not a corner (i.e., $p = E_i \cap E_j$ — irred. comp. of $\pi^{-1}(0)$) and not a saddle node with strong separatrix in $\pi^{-1}(0)$ ("bad" saddle-node).



3) Introduce the CAMACHO-SAD index $CS(p, E, \hat{\mathcal{F}}_\pi)$, where $p \in E \subseteq \pi^{-1}(0)$, generalizing the index $\alpha = \frac{\lambda_2}{\lambda_1}$ of an elementary singularity.

Local-global theorem: $\sum_{p \in E} CS(p, E) = E \cdot E$.

4) Combinatorial arguments involving the reduction process and the index formula.

Camacho-Sad index.

We start from a foliation generated by $\omega = f\,dx + g\,dy$, and tangent to the curve $C = \{y=0\}$.

The tangency condition translates into $y \mid f$.

As an example, take the linear foliation induced by $\omega = x\,dy - \alpha y\,dx$, $\alpha \in \mathbb{C}^*$

We have seen the importance of the holonomy h along the curve $(\underbrace{e^{2\pi i t}, 0}_{\gamma(t)})$.

With respect to the transverse $x = \text{const}$, $(\gamma, 0)$ lifts to

$\Gamma(t, y_0) = (e^{2\pi i t}; e^{2\pi i \alpha t} y_0 + \text{hot}(y_0))$ linear case: $\Gamma(t, y_0) = (e^{2\pi i t}, e^{2\pi i \alpha t} y_0)$

In particular, the holonomy is $h(y_0) = e^{2\pi i \alpha} y_0 + \text{h.o.t.}$

While $e^{2\pi i \alpha}$ describe (the first order of) how the leaves change after one turn around the singularity $x=0$ in C, its logarithm α describes also how many times the curve $\Gamma(t, y_0)$ winds around $y=0$.

The lift Γ has been computed by solving the equation $\frac{dy}{y} = -\frac{f'}{g}dx$, where $f = y f'$.

In the linear case, we get $\frac{dy}{y} = \frac{\alpha\,dx}{x}$, and $\alpha = \operatorname{Res}_0\left(\frac{\alpha\,dx}{x}\right)$

This leads to the following definition.

Def: The Camacho-Sad index of $\mathcal{F}$ induced by $\omega = y f' dx + g\,dy$ with respect to $C = \{y=0\}$ at a point $p \in C$ is: $CS(p, C, \mathcal{F}) := \operatorname{res}_p\left(-\frac{f'}{g}dx\Big|_{y=0}\right)$.

Rem: one can check directly that $CS(p, C, \mathcal{F})$ does not depend on the representant ω chosen, nor on the coordinates (adapted to C).

By a direct computation, for reduced singularities we get: $C_1 = \{y=0\}$, $C_2 = \{x=0\}$.

- regular point: $\omega = dy$ $\quad CS(0, C_1) = \operatorname{res}_0(0) = 0$.
- non-resonant Poincaré, or Siegel: $\omega = \lambda_1 x(1+o(1))dy - \lambda_2 y(1+o(1))dx$.

  $CS(0, C_1) = \operatorname{res}_0\left(\frac{\lambda_2}{\lambda_1 x}(1+o(1))dx\right) = \frac{\lambda_2}{\lambda_1} =: \alpha \qquad CS(0, C_2) = \frac{1}{\alpha}$.
- saddle-node: $\omega = x\,dy - y^{k+1}(1+\beta y^k)dx \; (+ y^N R_N\,dy)$

  $CS(0, C_1) = \operatorname{res}_0\left(\frac{y^k(1+\beta y^k)dx}{x}\Big|_{y=0}\right) = 0 \qquad CS(0, C_2) \underset{\substack{\uparrow \\ (\text{if convergent})}}{=} \operatorname{res}_0\left(\frac{dy}{y^{k+1}(1+\beta y^k)}\Big|_{x=0}\right) = -\beta$

Camacho-Sad Index along reduced curves [Bru, §2.2] [Suw 98, §V] [Saito 80]

Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2,0)$ generated by the 1-form $\omega = f\,dx + g\,dy$.
Let $C = \{\phi = 0\}$ be a (reduced, not necessarily irreducible) invariant curve for $\mathcal{F}$ (i.e., a separatrix).

Lemma: the invariance condition is equivalent to the condition $\phi \mid d\phi \wedge \omega$.

Preuve: ($\Leftarrow$) at $p \in C$, we have $d\phi \wedge \omega(p) = 0$.
This means that $\omega(p)$ and $d\phi(p)$ are parallel $\forall p \in C$. If p is a smooth point of C, by reducibility of C we deduce that $df(p) \neq 0$, and $\omega(p) = \beta^{\in\mathbb{C}} \cdot df(p)$.
We deduce that C is ω-invariant.

($\Rightarrow$) $\frac{1}{\phi} d\phi \wedge \omega$ is holomorphic at $\{p \mid \phi(p) \neq 0\} = \mathbb{C}^2 \setminus C$.
At $p \in C$, we have that $\omega(p) = \beta(p) \cdot d\phi(p)$ for some $\beta(p) \in \mathbb{C}$, hence $d\phi \wedge \omega$ vanishes along C. By reducibility of C, we deduce that $\phi \mid d\phi \wedge \omega$.

Prop: $C = \quad 0$ is ω-invariant $\Leftrightarrow$ $\exists\, a, b$ functions (with isolated zeroes on C, i.e., $(a,\phi) = (b,\phi) = 1$), η 1-form, s.t. $b\omega = a\,d\phi + \phi\eta$.

Preuve: ($\Leftarrow$) If $\omega = \frac{a}{b} d\phi + \frac{\phi}{b}\eta \Rightarrow d\phi \wedge \omega = \phi \frac{d\phi}{b} \wedge \eta$ is a multiple of ϕ and we conclude by the lemma.

($\Rightarrow$) write $\omega = f\,dx + g\,dy$.

$$d\phi \wedge \omega = (\phi_x dx + \phi_y dy) \wedge (f\,dx + g\,dy) = (\phi_x g - \phi_y f)\, dx \wedge dy.$$

By the lemma, $\phi_x g - \phi_y f = \phi \cdot u$ for some holomorphic map u.

Set $b = \phi_x$. Then

$$b\omega = \phi_x f\,dx + \phi_x g\,dy = \underbrace{f}_{a}\underbrace{(\phi_x dx + \phi_y dy)}_{d\phi} + \underbrace{(\phi_x g - \phi_y f)\,dy}_{\phi \cdot \underbrace{u\,dy}_{\eta}}$$ $\square$

Rem: We could have written

$$\underbrace{\phi_y}_{b}\,\omega = \phi_y f\,dx + \phi_y g\,dy = \underbrace{g}_{a}(\phi_x dx + \phi_y dy) + \underbrace{(\phi_y f - \phi_x g)\,dx}_{-\phi \underbrace{u\,dx}_{\eta}}$$

In particular, the preparation is not unique.

Def. The Camacho-Sad index of $\mathcal{F}$ along $C=\{\phi=0\}$ at $p\in C$ is defined as $CS(p,C,\mathcal{F}) = \operatorname{res}_p\left(-\frac{1}{a}\eta|_C\right)$, where $b\omega = a\,d\phi + \phi\eta$

Rem: if $\omega = \underbrace{y}_{\phi}\underbrace{f'dx}_{\eta} + \underbrace{g}_{a}\underbrace{dy}_{d\phi}$, $C=\{y=0\}$ $\Rightarrow$ $CS(p,C,\mathcal{F}) = \operatorname{res}_p\left(-\frac{1}{a}\eta|_C\right)$
$= \operatorname{res}_p\left(-\frac{1}{g}\cdot f'dx|_C\right)$

Rem: If $C = C_1\cup\ldots\cup C_s$ $\Rightarrow$ $CS(p,C) = \sum_{j=1}^{s} CS(p,C_j)$.
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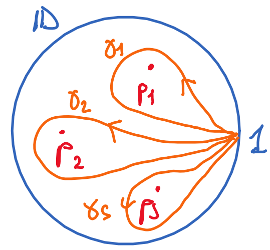
Globalization of the Camacho-Sad index. [Lor, §6.2]

Prop (Camacho-Sad index formula). Let E be an irreducible component of $\pi^{-1}(0)$. $\hat{\mathcal{F}}_\pi$ (saturated) foliation, E tangent to $\hat{\mathcal{F}}_\pi$.
Then $\sum_{p\in E} CS(p,E) = E\cdot E \quad \left(= c_1(N_{X_\pi/E}) \in \mathbb{Z}_{\leq -1}\right)$.

Idea of proof. Let $p_1\ldots p_s$ be the singularities of $\hat{\mathcal{F}}_\pi$. Up to change coordinates on $E\cong\mathbb{P}^1_{\mathbb{C}}$, we may assume $\{p_j\}\subset\mathbb{D}$.



Take γ_j loops at 1 in $\overline{\mathbb{D}}$ which do a positive turn around p_j,
and let $\gamma = \gamma_1 * \gamma_2 * \ldots * \gamma_s$ be their concatenation.
γ is homotopic to $\tilde\gamma(t) = e^{2\pi i t}$ the positive loop of $\partial\mathbb{D}$.
Since $\hat{\mathcal{F}}$ has no singularities on $E\setminus\mathbb{D}$, the holonomy along $\tilde\gamma$ is trivial, and we have that $\sum_{p_j} CS(p_j,E) =: \nu\in\mathbb{Z}$.

The lift of $\tilde\gamma$ to leaves is homotopic to $\tilde\Gamma(t) = \left(e^{2\pi i t}, y_0 e^{2\pi i\nu t}\right)$
Our goal is to construct a smooth perturbation E' of E, so that $E\cdot E = E\cdot E'$. On $E\setminus\mathbb{D}$, we pick the leaf of $\hat{\mathcal{F}}$ containing $(1,y_0)$ for $|y_0|<<1$.

Inside $\mathbb{D}$ we take $\begin{cases} (x, x^{\nu} y_0) & \text{if } \nu \geq 0 \\ (x, \bar{x}^{-\nu} y_0) & \text{if } \nu < 0 \end{cases}$ $(\star)$

This defines a perturbation of E that is piecewise smooth: we can deform it so to obtain another perturbation E' of E that is C^{∞}, and coincides with $(\star)$ on $\mathbb{D}_{\frac{1}{2}} = \{ |x| \leq \frac{1}{2} \}$, and $(E \setminus \mathbb{D}_{\frac{1}{2}}) \cap E' = \emptyset$.

But then $E \cap E' = \{0\}$, and $E \cdot E = E \cdot E' = \nu$ (the intersection being negative when $\nu < 0$ because $\bar{x}^{-\nu} \cdot y_0$ is anti-holomorphic: it reverses the orientation).

$\square$

Proof of Camacho-Sad theorem [J. Cano, 1997], see [IY, §14.E-F]

Camacho-Sad original argument is by contradiction, using the index theorem and the reduction process, in order to avoid corners.

Recall: E SNC divisor, E_j irred. comp. of E. $p \in E$ is called:

free if $\exists ! j$, $p \in E_j$

satellite, or corner, if $p = E_i \cap E_j$ $i \neq j$.

Later Cano simplifies the argument, making it constructive

Def Let X be a surface, $E \subset X$ a compact SNC curve, $\mathcal{F}$ a saturated foliation tangent to E.

We say that $p \in E$ is a Cano point if:

($\star$1) either $p \in E_1$ is free, and $CS(p, E_1) \notin \mathbb{Q}_{\geq 0}$

($\star$2) or $p = E_1 \cap E_2$ is satellite, and $\underset{i_1}{CS(p, E_1)} \in \mathbb{Q}_{<0}$, $\underset{i_2}{CS(p, E_2)} \notin \mathbb{Q}_{\geq \frac{1}{i_1}}$

(or the symmetric condition)

Rem: notice that if a Cano point is elementary, then

($\star$1) $\Rightarrow$ p is reduced not "bad" saddle-node $\Rightarrow$ $\exists$ separatrix transverse to E_1

($\star$2) $\Rightarrow$ p cannot be non-degenerate, because in that case $i_1 \cdot i_2 = 1$, and cannot be a saddle-node, since either $CS(p, E_1) = 0$ contradicts the first condition, or $CS(p, E_2) = 0$ $\Rightarrow$ $CS(p, E_1) < 0$, and $0 \geq \frac{1}{CS(p, E_1)}$, against the first condition.

Prop. Suppose $\mathcal{F}$ non-dicritic, $p \in \pi^{-1}(0)$ be a Camo point.
Let $\eta : \tilde{X} \to X$ be the blow-up of p, and $E = \eta^{-1}(p)$ be the exceptional divisor. Finally, let $\tilde{\mathcal{F}}$ be the saturated lift of $\mathcal{F}$.
Then at least a singularity of $\tilde{\mathcal{F}}$ in E is a Camo point.

Proof of Camacho-Sad theorem: Start from $(\mathbb{C}^2, 0)$, and blow-up the origin:
$\pi_0 : X_0 \to (\mathbb{C}^2, 0)$. $E_0 := \pi_0^{-1}(0)$ has self-intersection -1.
We deduce that $\mathcal{F}_0$ the saturated lift of $\mathcal{F}$ at X_0 has at least a singularity p_0 with $CS(p_0, E_0) \notin \mathbb{Q}_{\geq 0}$ (i.e., a Camo point).
We resolve the singularities of $\mathcal{F}$ (above p_0), and get a modification $\pi : X_\pi \to (\mathbb{C}^2, 0)$ dominating π_0, and a reduced Camo point above p_0.
By the remark above, $\mathcal{F}_\pi$ admits a separatrix $\tilde{C}$ transverse to $\pi^{-1}(0)$ at p.
Hence $C = \pi(\tilde{C})$ is a separatrix of $\mathcal{F}$. □

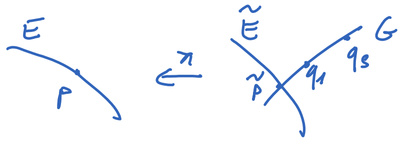
Before proving the prop., we need to understand how the index behaves under blow-up.
Lemma: $p \in E$, $\pi : \tilde{X} \to X$ blow-up of p. $G = \pi^{-1}(p)$, $\tilde{E} = \pi^* E$, $\tilde{p} = G \cap \tilde{E}$.
Then $CS(\tilde{p}, \tilde{E}) = CS(p, E) - 1$

Proof: this is a direct consequence of CS index formula, plus the fact that
$\tilde{E} \cdot \tilde{E} = E \cdot E - 1$

Alternatively, $\omega = y f' dx + g dy$ at p, where we picked coordinates (x,y) at p s.t. $E = \{y = 0\}$. Consider the blow-up $\pi(x,y) = (x, xy)$.
Then
$$\pi^* \omega = xy f'(x,xy) dx + g(x,xy)(x dy + y dx)$$
$$= y(x f'(x,xy) + g(x,xy)) dx + x g(x,xy) dy.$$
But then,
$$\underbrace{\mathrm{res}_{p'}\left(-\frac{x f' \circ \pi + g \circ \pi}{x g \circ \pi} dx \Big|_{y=0}\right)}_{CS(\tilde{p}, \tilde{E})} = \underbrace{\mathrm{res}_0\left(-\frac{f'(x)}{g(x)} dx\right)}_{CS(p,E)} + \underbrace{\mathrm{res}_0\left(-\frac{dx}{x}\right)}_{-1}$$
□

Proof of prop. ($\star 1$): Suppose first that $p \in E$ is a free Cano point.

We blow-up, and get



Suppose by contradiction that none of $\tilde{p}, q_1 \ldots q_s$ are Cano points.

$\Rightarrow \sum cs(q_j, G) \in \mathbb{Q}_{\geq 0}$

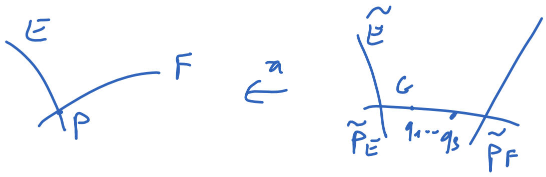
$\Rightarrow cs(\tilde{p}, G) = -1 - \sum cs(q_j, G) \in \mathbb{Q}_{\leq -1}$.

Since $\tilde{p}$ is not ($\star 2$), we must have $cs(\tilde{p}, \tilde{E}) \geq \underbrace{\frac{1}{cs(\tilde{p}, G)}}_{\in [-1, 0[}$
(and $\in \mathbb{Q}$)

But $cs(p, E) = cs(\tilde{p}, \tilde{E}) + 1 \in [0, +\infty)$, against the assumption ($\star 1$) for p.

($\star 2$) Suppose $p = E \cap F$ is a satellite Cano point, say with $cs(p, E) \in \mathbb{Q}_{<0}$.

We blow-up p, and get



As before, by contradiction, $\tilde{p}_E, \tilde{p}_F, q_1 \ldots q_s$ are not Cano.

Then $\sum_j cs(q_j, G) \in \mathbb{Q}_{\geq 0}$, and $cs(\tilde{p}_E, G) + cs(\tilde{p}_F, G) \in \mathbb{Q}_{\leq -1}$

we also have $\overbrace{cs(\tilde{p}_E, \tilde{E})}^{\tilde{\alpha}_E} = \overbrace{cs(p_E, E) - 1}^{\alpha_E - 1} \in \mathbb{Q}_{<-1}$, from which we deduce ($\tilde{p}_E$ not $\star 2$)

$cs(\tilde{p}_E, G) \in \mathbb{Q}_{\geq \frac{1}{\alpha_E - 1}}$. But then $cs(\tilde{p}_F, G) \in \mathbb{Q}_{\leq -1 + \frac{1}{1-\alpha_E}}$ $\overset{\tilde{p}_F \text{ not } \star 2}{\Longrightarrow}$ $\underbrace{cs(\tilde{p}_F, \tilde{F})}_{= \tilde{\alpha}_F} \in \mathbb{Q}_{\geq \frac{1}{cs(\tilde{p}_F, G)}}$

$-1 + \frac{1}{1-\alpha_E} = \frac{\alpha_E}{1-\alpha_E}$, $\quad \frac{1}{cs(\tilde{p}_F, G)} \geq \frac{1-\alpha_E}{\alpha_E}$

We deduce that $cs(p, F) =: \alpha_F = \tilde{\alpha}_F + 1 \geq \frac{1}{\alpha_E}$,

against the $\star 2$ assumption on p. $\square$

Related results [Pereira 2023 §4 and references therein]

[J. Cano, 1993]. Constructs a separatrix via Puiseux series: $C = \{(x, y(x^{\frac{1}{q}}))\}$,
must satisfy $f(x, y(x^{\frac{1}{q}})) + g(x, y(x^{\frac{1}{q}})) \cdot \frac{1}{q} x^{\frac{1}{q}-1} \cdot y'(x^{\frac{1}{q}}) = 0$

Techniques used are related to the Newton polygon associated to ω (to χ).

There are also estimates in the case s-Gevrey.

- Index techniques extend to ambient spaces that are singular, can give lower bounds on the number of separatrices depending on the dual graph of the singularity: [Camacho 1988], [Ortiz Bobadilla – Rosales González – Voronin 2010].

In higher dimension:

[Gomez Mont – Luengo, 1992] in $(\mathbb{C}^3, 0)$ vector fields might not have separatrices.

[Jouanolou §4, 1979] in $(\mathbb{C}^d, 0)$ $d \geq 3$, there are codim. 1 foliations without separatrices.

[Cano – Cerveau 1992 ($d=3$), Cano – Mattei 1992 ($d \geq 3$)] <u>non-dicritical</u> codim 1 foliations admit separatrices (= integral hypersurfaces)

(non-dicritical: in any model, or in a resolution, the saturated field is tangent to the exceptional divisor).